Robustness of Network Controllability with respect to Node removal

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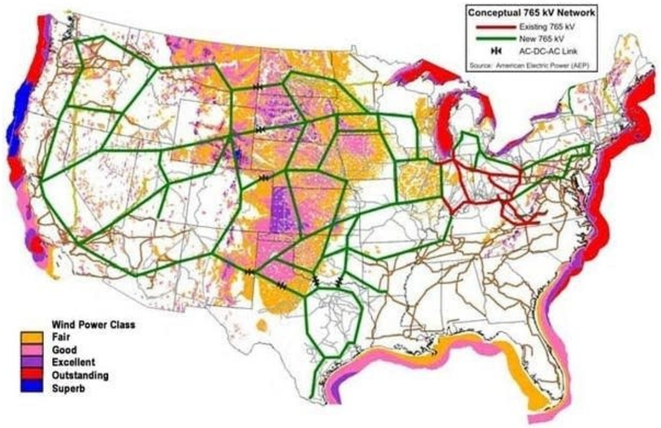
Outline

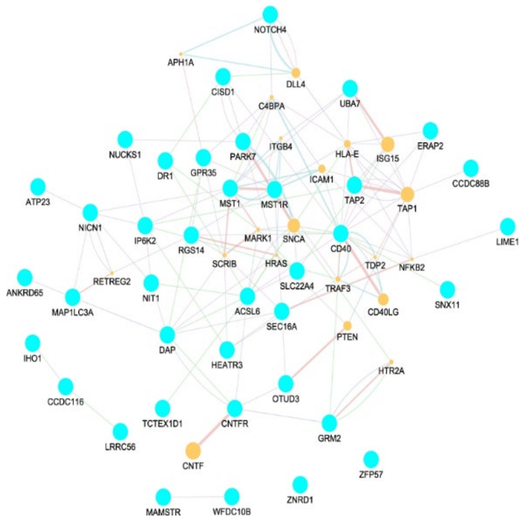
- Robustness metric: Network Controllability
- Node Removal
- Analytical methods
- Results
- Outlook

Transportation Networks



Power grids

Communication networks



Gene networks

Network controllability

Network is **controllable** if the states of nodes can be steered to any expected states in a finite time by imposing external inputs to some nodes (driver nodes).

How to judge whether a network is **controllable**?

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Kalman's rank condition:

LTI （linear time invariant system）

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

State vector $\boldsymbol{x} = (x_1, \ldots, x_N)^T$: states of nodes;

Coupling matrix A : $N \times N$, a_{ij} is the interaction strength from node j to node i;

Input vector $\mathbf{u} = (u_1, \ldots, u_m)^T$: m controllers;

Input matrix B: $N \times m$

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Kalman's rank condition:

$$\text{rank } C = \text{rank}[B, AB, A^2B, \dots, A^{N-1}B] = \mathrm{N}$$

Limitations:

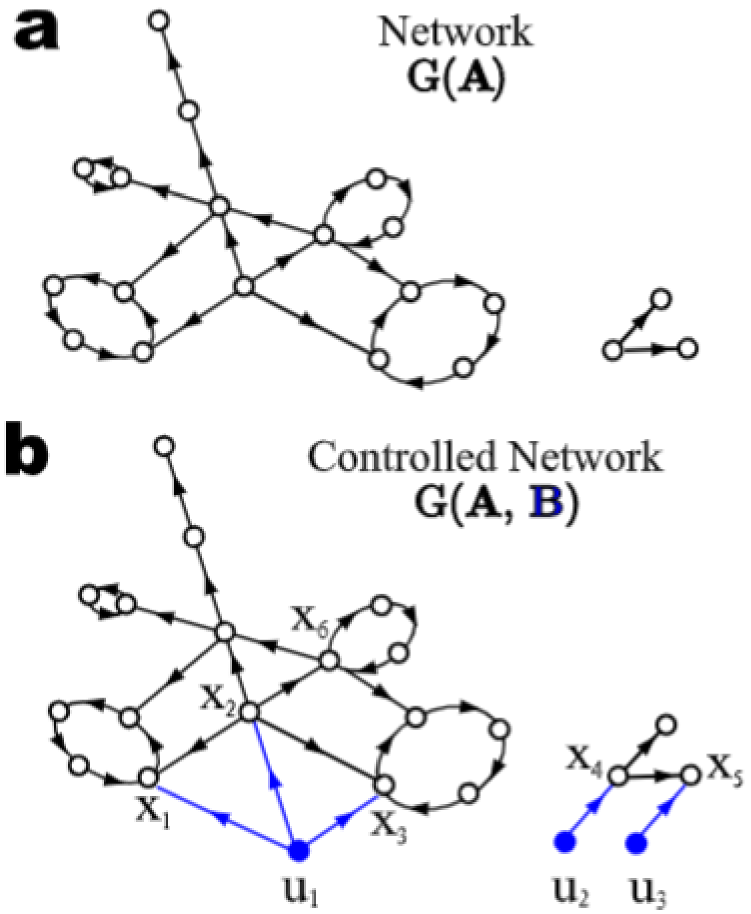
(1) Computationally expensive for large networks;

(2) Strong assumption of the interaction strength can be exactly measured.

[1] Kalman, Rudolf Emil. "Mathematical description of linear dynamical systems." *Journal of the Society for Industrial and Applied Mathematics, Series A: Control* 1.2 (1963): 152-192.
[2] Liu, Yang-Yu, and Albert-László Barabási. "Control principles of complex systems." *Reviews of Modern Physics* 88.3 (2016): 035006.

Structural network controllability (directed networks)

- **Graph interpretation:** The structured matrix $[A; B]$ is irreducible and has generic rank N
- **Method**: Maximum matching ; Minimum inputs theorem



Liu, Yang-Yu, Jean-Jacques Slotine, and Albert-László Barabási. "Controllability of complex networks." *Nature* 473.7346 (2011): 167-173.

Maximum Matching: an analytical method

$$G_{in}(x) = \sum_{k=0}^{\infty} P_{in}(k_{in}) x^{k_{in}},$$

$$G_{out}(x) = \sum_{k=0}^{\infty} P_{out}(k_{out}) x^{k_{out}},$$

$$H_{in}(x) = \frac{\sum_{k=1}^{\infty} k_{in} P_{in}(k_{in}) x^{k_{in}-1}}{< k_{in} >} = \frac{G'_{in}(x)}{G'_{in}(1)},$$

$$H_{out}(x) = \frac{\sum_{k=1}^{\infty} k_{out} P_{out}(k_{out}) x^{k_{out}-1}}{< k_{out} >} = \frac{G'_{out}(x)}{G'_{out}(1)},$$

$$n_d = \frac{1}{2}\{G_{in}(\omega_2) + G_{in}(1-\omega_1) - 2 + G_{out}(\hat{\omega}_2) + G_{out}(1-\hat{\omega}_1) + k[\hat{\omega}_1(1-\omega_2) + \omega_1(1-\hat{\omega}_2)]\},$$

where ω_1, ω_2, $\hat{\omega}_1$ and $\hat{\omega}_2$ satisfy

$$\omega_1 = H_{out}(\hat{\omega}_2),$$

$$\omega_2 = 1 - H_{out}(1-\hat{\omega}_1),$$

$$\hat{\omega}_1 = H_{in}(\omega_2),$$

$$\hat{\omega}_2 = 1 - H_{in}(1-\omega_1),$$

$$k = \tfrac{1}{2} < k > = < k_{in} > = < k_{out} >.$$

$$\boxed{n_D = n_d(1-p) + p}$$

p is the fraction of removed nodes

Liu, Yang-Yu, Jean-Jacques Slotine, and Albert-László Barabási. "Controllability of complex networks." *Nature* 473.7346 (2011): 167-173.

Node Removal

- **Random node removal**

  Each node i has the same probability p to be removed;

- **Target node removal**

  The removed probability p_i of node i is based on the total degree k_i

$$p_i = \frac{{k_i}^{\alpha}}{\sum_{j \in \mathrm{N}} {k_j}^{\alpha}}$$

two cases:

$\alpha = 1$ (larger degree, larger probability to be removed);

$\alpha = 10$ (close to remove the node with the largest degree).

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Networks

- **Synthetic directed networks**: ER networks, Swarm Signaling networks (SSNs)
- **Real networks**:

  Four communication networks

Name	N	L	$< k >$
HinerniaGlobal	55	81	2.95
Deltacom	113	161	2.85
GtsCe	149	193	2.59
Cogentco	197	243	2.49

Table 1: Properties of four real-world communication networks

https://networks.skewed.de/net/internet_top_pop

The analytical approximations

- Random removal

$$\boxed{\begin{aligned}\bar{G}_{in}(x) &= G_{in}(p+(1-p)x),\\ \bar{G}_{out}(x) &= G_{out}(p+(1-p)x),\end{aligned}}$$

$$\bar{H}_{in}(x) = \frac{\bar{G}'_{in}(x)}{\bar{G}'_{in}(1)},$$

$$\bar{H}_{out}(x) = \frac{\bar{G}'_{out}(x)}{\bar{G}'_{out}(1)}.$$

$$n_D = \frac{1}{2}(1-p)\{\bar{G}_{in}(\omega_2) + \bar{G}_{in}(1-\omega_1) - 2 + \bar{G}_{out}(\hat{\omega_2}) + \bar{G}_{out}(1-\hat{\omega_1}) + k(1-p)[\hat{\omega_1}(1-\omega_2) + \omega_1(1-\hat{\omega_2})]\} + p,$$

Shao, Jia, Sergey V. Buldyrev, Lidia A. Braunstein, Shlomo Havlin, and H. Eugene Stanley. "Structure of shells in complex networks." *Physical Review E* 80, no. 3 (2009): 036105.

The analytical approximations

- Random removal (synthetic networks)

ER networks:

$$G_{in}(x) = e^{-k(-x+1)}, \quad G_{out}(x) = e^{-k(-x+1)},$$

$$n_D = p + p\omega_2 - \omega_2 + [1 - p + k(1-p)^2(1-\omega_2)]e^{k(1-p)(\omega_2-1)}$$

$$\text{where } \omega_2 \text{ satisfies } 1 - \omega_2 - e^{-k(1-p)e^{-k(1-p)(1-\omega_2)}} = 0.$$

SSNs:

$$G_{in}(x) = e^{-k(-x+1)}, \quad G_{out}(x) = x^k.$$

$$n_D = p + p\omega_2 - \omega_2 + [1 - p + (k-1)(1-p)^2(1-\omega_2)]e^{k(1-p)(\omega_2-1)}$$

$$\text{where } \omega_2 \text{ satisfies } 1 - \omega_2 - [p + (1-p)(1 - e^{-k(1-p)(1-\omega_2)}]^{k-1} = 0.$$

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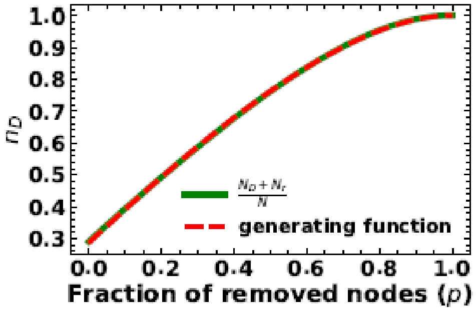
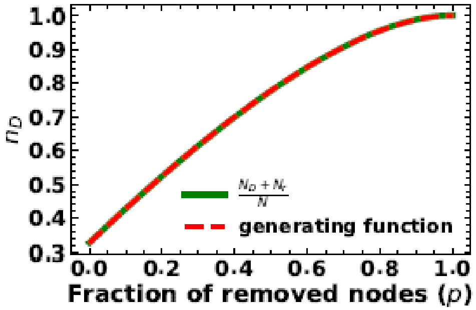
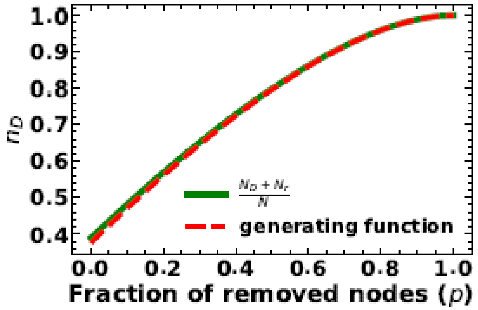
Results of random node removal



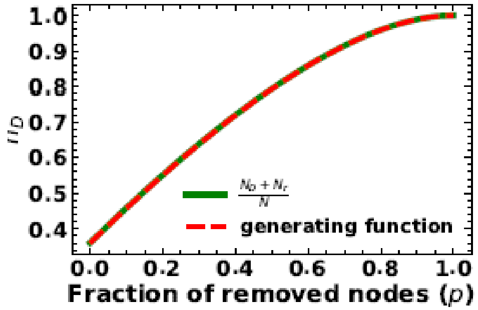
(a) HinerniaGlobal



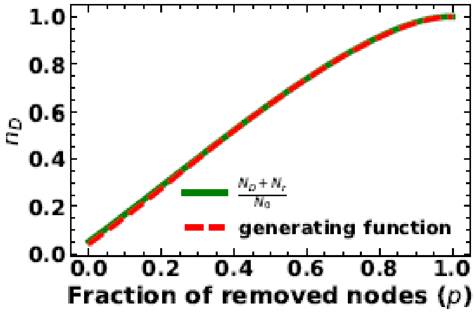
(b) Deltacom



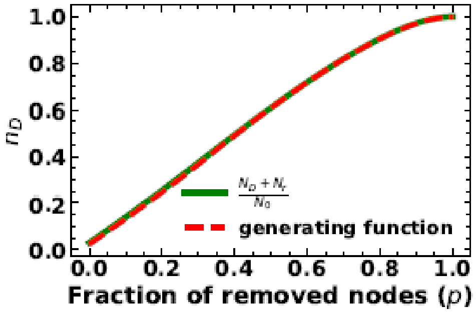
(c) GtsCe



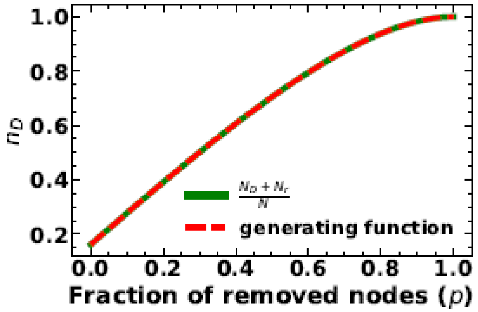
(d) Cogentco



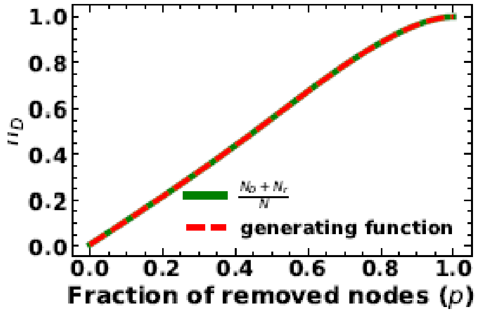
(e) ER(50,0.07)



(f) ER(100,0.04)



(g) SSN(10^4, 2)



(h) SSN(10^4, 5)

The analytical approximations

Target removal p → **Random removal $\bar{p}$**

- Target removal ($\alpha = 1$)

$$p_i = \frac{k_i}{\sum_{j \in N} k_j}$$

$$\boxed{\bar{p} = 1 - \frac{f G'_\alpha(f)}{< k >},}$$

where $f \equiv G_\alpha^{-1}(1-p)$, $G_\alpha(x) \equiv \sum_k p_k x^{k^\alpha}$

$< k >$ is the average total degree

ER networks:

$$G(x) = e^{-<k>(-x+1)}$$

SSNs:

$$G(x) = x^{\frac{<k>}{2}} e^{-\frac{<k>}{2}(-x+1)}$$

Kenett D Y, Gao J, Huang X, et al. Network of interdependent networks: overview of theory and applications[J]. Networks of Networks: The Last Frontier of Complexity, 2014: 3-36.

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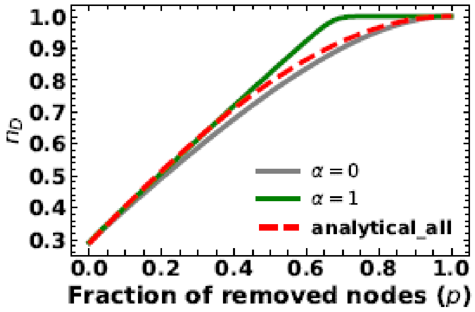
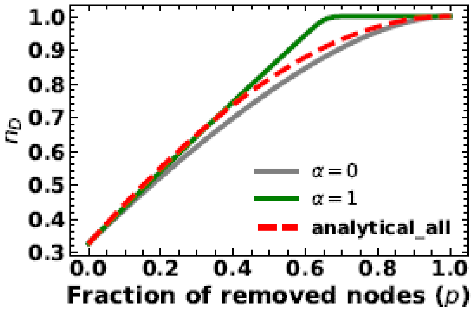
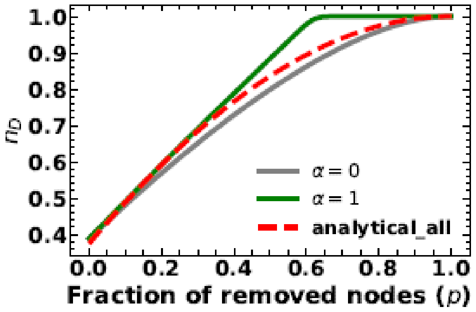
Results of target node removals ($\alpha = 1$)



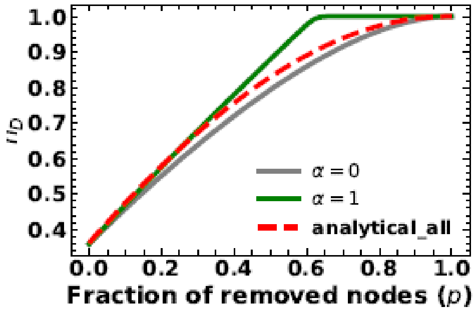
(a) HinerniaGlobal



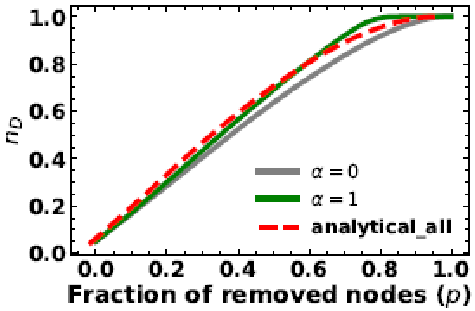
(b) Deltacom



(c) GtsCe



(d) Cogentco



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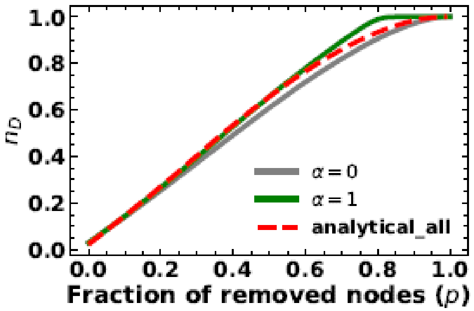

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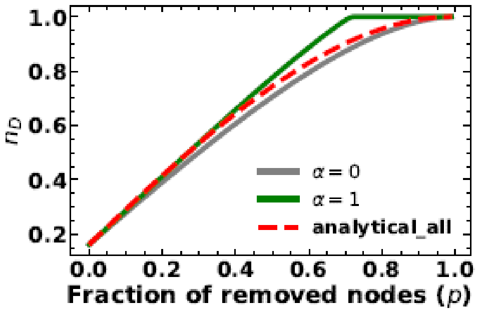

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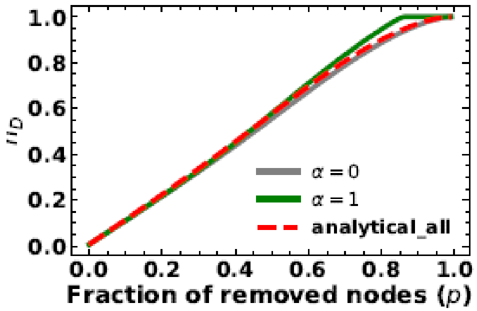

(h) SSN(10^4, 5)

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The analytical approximations

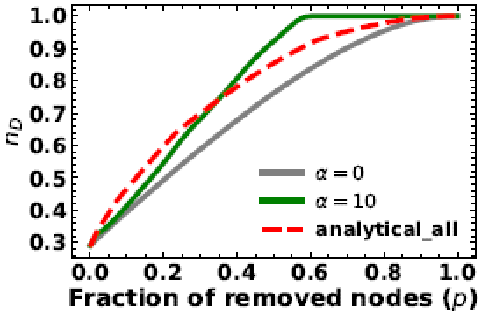
Target removal p → **Random removal $\bar{p}$**

- Target removal ($\alpha = 10$)

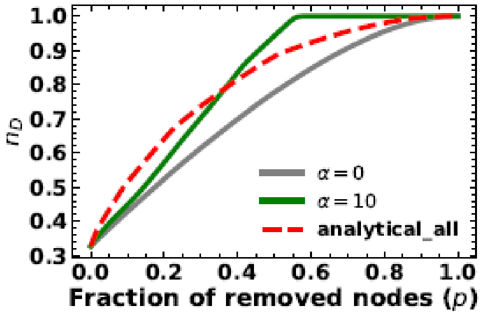
$$p_i = \frac{{k_i}^{10}}{\sum_{j \in N} {k_j}^{10}}$$

$$\boxed{\bar{p}} = \frac{\sum_{k=k_{max}}^{k=\bar{k}} p_k N k}{N<k>} = \frac{\sum_{k=k_{max}}^{k=\bar{k}} p_k k}{<k>}$$

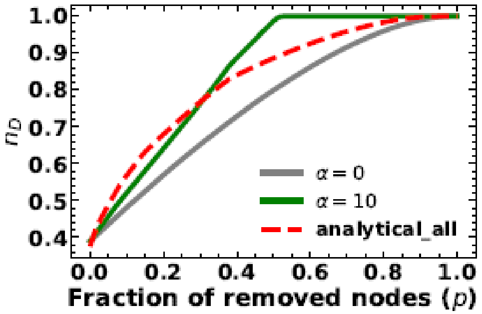
Results of target node removals ($\alpha = 10$)



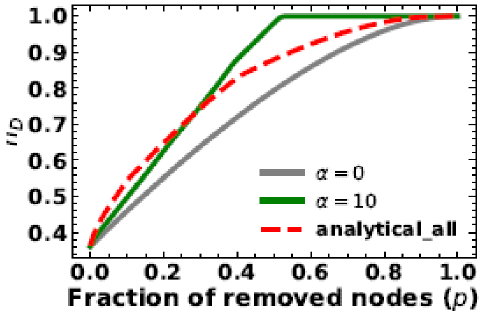
(a) HinerniaGlobal



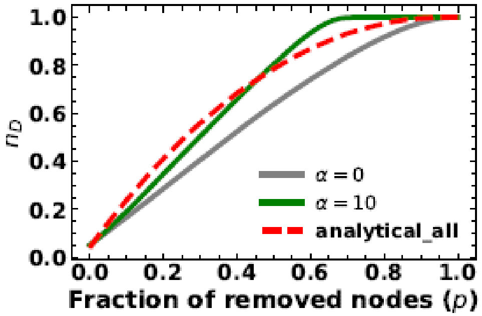
(b) Deltacom



(c) GtsCe



(d) Cogentco



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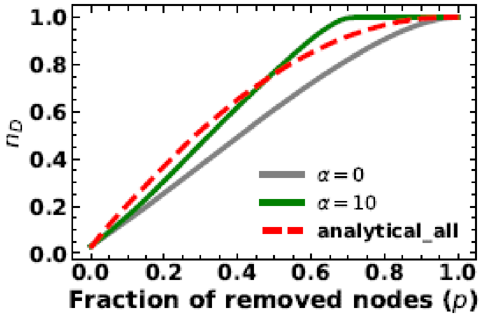

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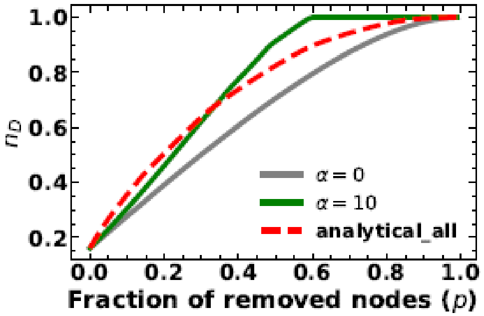

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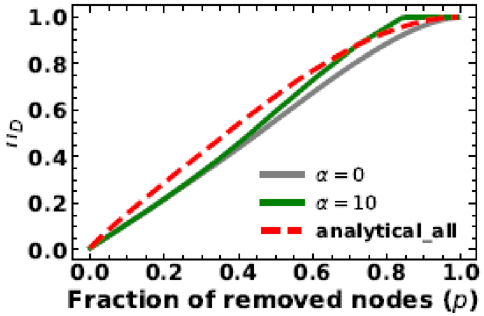

(h) SSN(10^4, 5)

Outlook

- Other kinds of target node removal based on in-degree and out-degree
- Other synthetic networks: scale-free networks, small networks
- Recoverability of network controllability with respect to node addition

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Thank you for listening!

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